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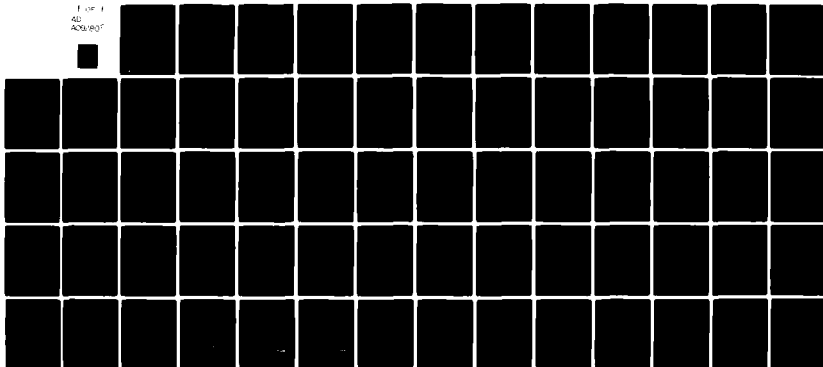
DIRECTORATE OF AEROSPACE STUDIES KIRTLAND AFB NM
METHODODOGY FOR THE EFFECTIVENESS EVALUATION OF STAND-OFF ATTAC--ETC(U)
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**METHODOLOGY FOR THE EFFECTIVENESS
EVALUATION OF STAND-OFF ATTACK
ON MOVING TANK COLUMNS**

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**DIRECTORATE OF AEROSPACE STUDIES
DCS/PLANS AND PROGRAMS, HQ AFSC
KIRTLAND AFB, NEW MEXICO 87117**

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This report documents some of the methodology that was generated in support of the study "Stand-Off Applications of Wide Area Anti-Armor Munitions (WAAM)." This study was conducted for Headquarters Air Force Systems Command by the Directorate of Aerospace Studies at Kirtland Air Force Base in 1978/79.

The author is indebted to Courtlandt Bohn for critically reviewing the manuscript and offering valuable suggestions, to Hans J. Gevelhoff for suggesting the multiple columns analysis, and to A. Foster Cooper and Christopher Feuchter for computerizing the theory, thus demonstrating its feasibility.

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The analysis operates with a "velocity distribution function" which may be symmetric (Gaussian) or asymmetric (non-Gaussian). The constitutive parameters of this function are one or two characteristic velocities (such as the "mean velocity") and one or two standard deviations, depending on the symmetry properties. These parameters incorporate the information which is available about the motion of a hostile tank column toward the FEBA.

On the basis of the velocity distribution function and other, more conventional inputs such as the impact distribution function and the impact point related kill probability, the expected number of tanks destroyed is established as an analytical function of the following input data: Number of tanks and length of the column, original target location error by reconnaissance, characteristic velocities and standard deviations of the velocity distribution function, delay time between reconnaissance and strike, aimpoint selection, delivery accuracies, and impact point related kill probability.

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LIST OF SYMBOLS

		SECTION WHERE SYMBOLS DEFINED
A	Aimpoint	I
A*	Optimal Aimpoint	L
$f(v)$	Velocity Distribution Function	B
$f_1(v)$	Velocity Distribution Function	C
$f_2(v)$	Velocity Distribution Function	C
$f_x(v_x)$	Velocity Distribution Function for X-Direction	O
$f_y(v_y)$	Velocity Distribution Function for Y-Direction	O
$f_r(v_r)$	Relative Velocity Distribution Function	M
$F(s)$	Spacing Distribution Function	M
$F_R(s)$	Renormalized Spacing Distribution Function	N
$g(x)$	Position Distribution Function	E
$g_1(x)$	Position Distribution Function	F
$g_2(x)$	Position Distribution Function	F
$G_x(x)$		O
$G_y(y)$		O
$h(I/A)$	Impact Distribution Function	J
I	Impact Point	I
L	Half-Length of Tank Column	G
M	Number of Columns per Aggregate	N
m	Number of Bombs	K
N	Number of Tanks per Column	H
n	Number of Tanks per Unit Length	H
$P_o(x_t)$	Probability of Occupancy	G
$P_o(x_t/s)$	Conditional Probability of Occupancy	N
$P_m(x_t)$	Probability of Occupancy after m Bombings	K

$P_K(T/I)$	Impact Point Related Kill Probability	J
$P_K(T/A)$	Aimpoint Related Kill Probability	J
s	Spacing Between Columns	M
s_0	Spacing at Reconnaissance Time	M
t_r	Reconnaissance Time	A
t	Strike Time	A
T	Contemplated Target Point	A
v	Column Velocity	B
\bar{v}	Most Probable Value of v	B
$\langle v \rangle$	Expected Value of v	B
$\langle v^2 \rangle$	Expected Value of v^2	B
v_r	Relative Velocity Between Columns	M
x_a	Coordinate of A	I
x_i	Coordinate of I	I
x_t	Coordinate of T	I
β	Attack Angle	I
$v_0(x_t)$	Exterior Tank Density	H
$v_m(x_t)$	Exterior Tank Density After m Bombs	K
σ_v	Standard Deviation of Velocity Distribution Function	B
σ_1	Standard Deviation	C
σ_2	Standard Deviation	C
σ_0	Standard Deviation Associated with Target Location Error	E
σ_p	Standard Deviation of Position Distribution Function	E
σ_{vp}	Standard Deviation of Velocity Induced Position Distribution Function	E
σ_r	Standard Deviation of Relative Velocity Distribution Function	M
σ_s	Standard Deviation of Spacing Distribution Function	M

A. INTRODUCTION

The purpose of stand-off is survivability enhancement. The price to pay is strike delay. For moving targets, strike delay means effectiveness reduction. Therefore, the effectiveness analysis of stand-off attack on a moving tank column must assess the effectiveness reduction by strike delay. This is the prime subject of the present methodology study.

The study addresses the following scenario. A tank column is on the road march behind the FEBA (Forward Edge of the Battle Area). Most likely, but not necessarily, the column moves toward the FEBA. Blue has obtained reconnaissance about this fact and plans a stand-off airstrike or missile-strike.

We use the (straight) road as x-axis and place the FEBA in the positive x-direction. This has the effect that the mean column velocity is positive.

The information available to Blue may now be described as follows:

1. At "reconnaissance time"

$$t_r = 0 \quad (1-1)$$

the middle point of the tank column occupied the position

$$x = 0 \quad (1-2)$$

2. The average travel velocity of a hostile tank column on the road march to the FEBA is $\langle v \rangle$. It is assumed that

$$\langle v \rangle > 0 \quad (1-3)$$

3. The length of the column is $2L$.

4. The strike takes place at the "strike time"

$$t \geq 0 \quad (1-4)$$

The methodology offered in this paper is designed to answer questions of the following type.

1. What is the probability that, at strike time t , the "contemplated target point T " with the coordinate x_t is occupied by the tank column?
2. What is the expected number of tanks destroyed if, at strike time t , a bomb is aimed at the aimpoint A with the coordinate x_a ?
3. What is the expected number of targets destroyed if, at strike time t , m bombs are aimed at the aimpoints A_1, \dots, A_m with the coordinates x_1, \dots, x_m ?
4. What selection of aimpoints maximizes the expected number of tanks destroyed?
5. What are the answers to the aforementioned questions if there is not one tank column, but several?

These questions illustrate only the main thrust of the present methodology study, but there will be excursions and extensions, for example, a two-dimensional tank array off the road, an asymmetric velocity distribution function, and others.

B. SYMMETRIC VELOCITY DISTRIBUTION FUNCTION

The chief instrument of the analysis outlined in the previous section is the "Velocity Distribution Function" (VDF). In the present section, we assume that the VDF is symmetric with respect to the "most probable velocity" \bar{v} , also called "mode". If nothing else is known about the velocity distribution, then it follows in conjunction with the Central Limit Theorem of the calculus of probability that the VDF is Gaussian. If σ_v denotes the "standard deviation", we have

$$f(v) = \frac{1}{\sigma_v \sqrt{2\pi}} \exp \left\{ -\frac{(v-\bar{v})^2}{2 \sigma_v^2} \right\} \quad (2-1)$$

The Gaussian VDF is constituted by two parameters: \bar{v} and σ_v . These parameters can be interpreted in terms of the "mean velocity" $\langle v \rangle$ and the "mean square velocity" $\langle v^2 \rangle$. To carry this out, we consider the integral equations

$$\int_{-\infty}^{+\infty} f(v) dv = 1 \quad (2-2a)$$

$$\int_{-\infty}^{+\infty} f(v) v dv = \langle v \rangle \quad (2-2b)$$

$$\int_{-\infty}^{+\infty} f(v) v^2 dv = \langle v^2 \rangle \quad (2-2c)$$

Here, equation (2a) is simply the "normalization condition" which is satisfied identically. The two remaining equations yield

$$\bar{v} = \langle v \rangle \quad (2-3a)$$

$$\sigma_v^2 = \langle v^2 \rangle - \langle v \rangle^2 \quad (2-3b)$$

Hence the most probable and the mean velocity are identical. This, of course, is a consequence of the symmetry of the VDF.

Let us now assume that a large number, say N , observational or intelligence data are available about the motion of hostile tank columns on the road march to the FEBA. Let v_i denote the travel velocity of the i th observation. We then have

$$\langle v \rangle = \frac{1}{N} \sum_{i=1}^N v_i \quad (2-4a)$$

$$\langle v^2 \rangle = \frac{1}{N} \sum_{i=1}^N v_i^2 \quad (2-4b)$$

If these equations are heeded in relations (3a) and (3b), we see that the constitutive parameters \bar{v} and σ_v can be interpreted in terms of the intelligence data v_i .

An example will illustrate this. We assume that the following intelligence information is available.

1. At any given time, 80% of the columns move with approximately the same speed of 30 km/h.
2. At any given time, 20% of the columns are at rest.

We then have (with the same approximation):

Moving: $N_1 = 0.8N$, $v_i = 30$ km/h

At Rest: $N_0 = 0.2N$, $v_i = 0$

This yields

$$\bar{v} = \langle v \rangle = 24 \text{ km/h}$$

$$\sigma_v = 12 \text{ km/h}$$

C. ASYMMETRIC VELOCITY DISTRIBUTION FUNCTION

Whether or not the VDF is symmetric depends on the information which is available with regard to the travel velocities of hostile tank columns. We shall elaborate on this subject in section D. In the present section, we simply assume that the information available dictates an asymmetric VDF. The question then is how to construct this function.

Probably the most elegant way is the following. We take two Gaussian functions

$$f_1(v) = \frac{1}{\sigma_1 \sqrt{2\pi}} \exp \left\{ -\frac{(v-\bar{v})^2}{2\sigma_1^2} \right\} \quad (3-1a)$$

$$f_2(v) = \frac{1}{\sigma_2 \sqrt{2\pi}} \exp \left\{ -\frac{(v-\bar{v})^2}{2\sigma_2^2} \right\} \quad (3-1b)$$

They have the same mode, \bar{v} , but different standard deviations σ_1 and σ_2 . For our present purposes, we assume

$$\sigma_1 < \sigma_2 \quad (3-2)$$

As in the previous section, we denote the actual VDF by $f(v)$, but it must now be kept in mind that $f(v)$ is asymmetric. The idea is to "compose" $f(v)$ by $f_1(v)$ and $f_2(v)$. For $v < \bar{v}$, $f(v)$ shall be equal to $f_1(v)$, except for a constant factor still to be determined. For $v > \bar{v}$, $f(v)$ shall be equal to $f_2(v)$, except for another factor.

The purpose of the two factors is two-fold. First, at $v = \bar{v}$, the two functions must be fused together. And since

$$f_1(\bar{v}) = \frac{1}{\sigma_1 \sqrt{2\pi}} \quad (3-3a)$$

$$f_2(\bar{v}) = \frac{1}{\sigma_2 \sqrt{2\pi}}$$

it follows that equalization is achieved if $f_1(v)$ is multiplied by σ_1 , and $f_2(v)$ is multiplied by σ_2 .

Second, the actual VDF, $f(v)$, must be normalized. Therefore, we need one free parameter, say σ_v , to achieve normalization. If we then write the two factors in the form σ_1/σ_v and σ_2/σ_v , the asymmetric VDF assumes the following form

$$f(v) = \begin{cases} \frac{\sigma_1}{\sigma_v} f_1(v) & v \leq \bar{v} \\ \frac{\sigma_2}{\sigma_v} f_2(v) & v \geq \bar{v} \end{cases} \quad (3-3b)$$

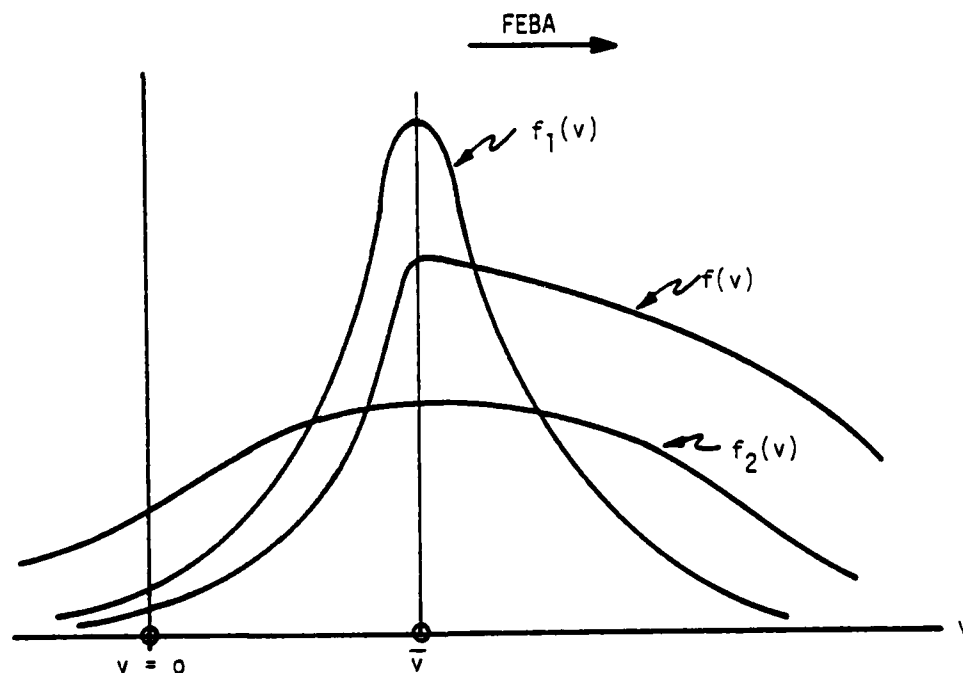


Figure 1. Asymmetric Velocity Distribution Function and its Constituents

The three functions $f_1(v)$, $f_2(v)$, and $f(v)$ are illustrated in figure 1. It can be seen that $f(v)$ is "biased" in favor of positive velocity values. This, of course, is a consequence of assumption (3-2) which, in turn, accounts for our earlier assumption (section 8) that the FEBA lies in positive x-direction.

We intimated already that $f(v)$ is not yet normalized, but that normalization can be achieved by proper selection of the parameter σ_v . The normalization condition has the form of equation (2-2a). If these equations (3) are heeded, we obtain

$$\frac{\sigma_1}{\sigma_v} \int_{-\infty}^{\bar{v}} f_1(v) dv + \frac{\sigma_2}{\sigma_v} \int_{\bar{v}}^{+\infty} f_2(v) dv = 1 \quad (3-4)$$

Here we observe that $f_1(v)$ and $f_2(v)$ are normalized and that they are symmetric with respect to the axis $v=\bar{v}$. Hence it follows that

$$\frac{\sigma_1}{2\sigma_v} + \frac{\sigma_2}{2\sigma_v} = 1$$

or

$$\sigma_v = \frac{1}{2} (\sigma_1 + \sigma_2) \quad (3-5)$$

Relation (5) is necessary and sufficient for the normalization of the asymmetric VDF, $f(v)$. The relation also demonstrates that only two of the three standard deviations σ_v , σ_1 , σ_2 are freely selectable. If we chose, for example, σ_1 and σ_2 as freely selectable, then $f(v)$ is constituted by three parameters: \bar{v} , σ_1 , σ_2 . Therefore, anybody who claims that the VDF is asymmetric should be able to prove this by demonstrating that the VDF, indeed, is constituted by three distinct and freely selectable parameters. We shall elaborate on this subject in the following section.

D. RELATIONS BETWEEN THE CONSTITUTIVE PARAMETERS OF THE ASYMMETRIC VELOCITY DISTRIBUTION FUNCTION

Probabilities and distribution functions are based on information. If the information changes, so do the probabilities and the distribution functions. The information content of a distribution function is stored in its constitutive parameters. However, in practical situations such as stand-off attack on a moving tank column, the information is rarely stored in such highly theoretical parameters as σ_v , σ_1 , σ_2 , but most likely in more observable parameters such as \bar{v} , $\langle v \rangle$, $\langle v^2 \rangle$. Hence it is desirable, and it will also be instructive, to establish σ_v , σ_1 , and σ_2 as functions of \bar{v} , $\langle v \rangle$, and $\langle v^2 \rangle$. This is the purpose of the present section.

The mean values $\langle v \rangle$ and $\langle v^2 \rangle$ are defined by equations (2-2b) and (2-2c). If there we substitute relation (3-3), we obtain

$$\langle v \rangle = I_1 + I_2 \quad (4-1a)$$

$$\langle v^2 \rangle = J_1 + J_2 \quad (4-1b)$$

with

$$I_1 = \frac{\sigma_1}{\sigma_v} \int_{-\infty}^{\bar{v}} f_1(v) v dv \quad (4-2a)$$

$$I_2 = \frac{\sigma_2}{\sigma_v} \int_{\bar{v}}^{+\infty} f_2(v) v dv \quad (4-2b)$$

$$J_1 = \frac{\sigma_1}{\sigma_v} \int_{-\infty}^{\bar{v}} f_1(v) v^2 dv \quad (4-2c)$$

$$J_2 = \frac{\sigma_2}{\sigma_v} \int_{\bar{v}}^{+\infty} f_2(v) v^2 dv \quad (4-2d)$$

The evaluation of the integrals I_1, I_2, J_1, J_2 is straightforward but tedious. It will be facilitated by the following auxiliary formulae. First, from equations (3-1a) and (3-1b), it follows by differentiation that

$$vf_k(v) = \bar{v} f_k(v) - \sigma_k^2 \frac{\partial f_k}{\partial v} \quad (k=1,2) \quad (4-3a)$$

This may be written in the form

$$v^2 f_k = \bar{v}^2 f_k - (v + \bar{v}) \sigma_k^2 \frac{\partial f_k}{\partial v} \quad (k=1,2) \quad (4-3b)$$

For this, we may also write

$$v^2 f_k = \bar{v}^2 f_k - \bar{v} \sigma_k^2 \frac{\partial f_k}{\partial v} - \sigma_k^2 \left\{ \frac{\partial}{\partial v} (vf_k) - f_k \right\} \quad (k=1,2) \quad (4-3c)$$

We also mobilize the following formulae. Defining

$$E(x) = \frac{1}{\lambda \sqrt{2\pi}} \exp \left\{ -\frac{x^2}{\lambda^2} \right\} \quad (4-4a)$$

and

$$\phi_m = \int_0^\infty E(x) x^m dx \quad (4-4b)$$

We have

$$\phi_0 = \frac{1}{2} \quad (4-5a)$$

$$\phi_1 = \frac{\lambda}{2\pi} \quad (4-5b)$$

$$\phi_2 = \frac{1}{2} \lambda^2 \quad (4-5c)$$

With the aid of equations (4-3a), (4-3b), (4-3c), (4-4a), (4-4b), (4-5a), (4-5b), and (4-5c), the integrals I_1 , I_2 , J_1 , and J_2 can now be solved. The results are

$$I_1 = \frac{\sigma_1}{\sigma_v} \left\{ \frac{\bar{v}}{2} - \frac{\sigma_1}{\sqrt{2\pi}} \right\} \quad (4-6a)$$

$$I_2 = \frac{\sigma_2}{\sigma_v} \left\{ \frac{\bar{v}}{2} + \frac{\sigma_2}{\sqrt{2\pi}} \right\} \quad (4-6b)$$

$$J_1 = \frac{1}{2\sigma_v} \left\{ \sigma_1 \bar{v}^2 - \frac{4\sigma_1^2 \bar{v}}{\sqrt{2\pi}} + \sigma_1^3 \right\} \quad (4-7a)$$

$$J_2 = \frac{1}{2\sigma_v} \left\{ \sigma_2 \bar{v}^2 + \frac{4\sigma_2^2 \bar{v}}{\sqrt{2\pi}} + \sigma_2^3 \right\} \quad (4-7b)$$

Now substituting equations (4-6a), (4-6b), (4-7a), and (4-7b) into equations (4-1a) and (4-1b), we get

$$\langle v \rangle = \frac{\bar{v}(\sigma_1 + \sigma_2)}{2\sigma_v} + \frac{\sigma_2^2 - \sigma_1^2}{\sigma_v \sqrt{2\pi}} \quad (4-8)$$

and

$$\langle v^2 \rangle = \frac{1}{2\sigma_v} \cdot \left\{ \sigma_1^3 + \sigma_2^3 + \frac{4\bar{v}(\sigma_2^2 - \sigma_1^2)}{\sqrt{2\pi}} + \bar{v}^2 (\sigma_1 + \sigma_2) \right\} \quad (4-9)$$

Under observance of equation (3-5) and the identity

$$\frac{\sigma_1^3 + \sigma_2^3}{\sigma_1 + \sigma_2} = \sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2 \quad (4-10)$$

equations (4-8) and (4-9) assume the form

$$\langle v \rangle = \bar{v} + \sqrt{\frac{2}{\pi}} (\sigma_2 - \sigma_1) \quad (4-11)$$

$$\langle v^2 \rangle = \bar{v}^2 + 2 \sqrt{\frac{2}{\pi}} \bar{v} (\sigma_2 - \sigma_1) + \sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2 \quad (4-12)$$

From equations (3-5) and (4-11), we also obtain the useful relations

$$\sigma_1 = \sigma_v - \sqrt{\frac{\pi}{8}} \left\{ \langle v \rangle - \bar{v} \right\} \quad (4-13a)$$

$$\sigma_2 = \sigma_v + \sqrt{\frac{\pi}{8}} \left\{ \langle v \rangle - \bar{v} \right\} \quad (4-13b)$$

For a brief discussion of these results, we first observe the following. From relations (3-2) and (4-11) follows that

$$\langle v \rangle > \bar{v} \quad (4-14)$$

In other words, the bias in favor of positive velocities places the mean velocity, $\langle v \rangle$, to the right of the most probable velocity, \bar{v} . This result follows also intuitively directly from figure 1.

For further discussion, we solve equations (4-11) and (4-12) for σ_1 and σ_2 . This leads to quadratic equations which have solutions of the form of equations (13a) and 13b). It is not difficult to show that

$$\sigma_v^2 = \langle v^2 \rangle - \langle v \rangle^2 + \left\{ 1 - \frac{3\pi}{8} \right\} \left\{ \langle v \rangle - \bar{v} \right\}^2 \quad (4-15a)$$

Equations (4-13a), (4-13b), and (4-15a) establish σ_1 , σ_2 , σ_v as functions of \bar{v} , $\langle v \rangle$, and $\langle v^2 \rangle$. This was the object of the present section. We observe that, for $\bar{v} = \langle v \rangle$, we have

$$\sigma_v = \sigma_1 = \sigma_2 = \sqrt{\langle v^2 \rangle - \langle v \rangle^2} \quad (4-15b)$$

Finally, we observe from relation (15a) that σ_v is real if, and only if,

$$\langle v^2 \rangle - \langle v \rangle^2 \geq \left\{ \frac{3\pi}{8} - 1 \right\} \left\{ \langle v \rangle - \bar{v} \right\}^2 \quad (4-16a)$$

This condition is more stringent than the condition

$$\langle v^2 \rangle - \langle v \rangle^2 \geq 0 \quad (4-16b)$$

which applies in the symmetric case.

In summary, we may now say the following. The VDF is asymmetric if, and only if, $\langle v \rangle \neq \bar{v}$. In particular, if $\langle v \rangle > \bar{v}$, the VDF is biased in favor of velocities greater than \bar{v} . It is frequently argued that we must assume this bias because of our earlier assumption that the column moves in the positive x-direction. However, this argument is faulty because the latter assumption has already been accounted for by the relation $\bar{v} > 0$. Therefore, the assumption $\langle v \rangle > \bar{v}$ constitutes an additional assumption which can only be justified on the basis of additional information. In other words, the symmetry assumption $\langle v \rangle = \bar{v}$ is the analytical expression of the absence of such additional information.

E. SYMMETRIC POSITION DISTRIBUTION FUNCTION

Let x denote the coordinate of the column's middle point at strike time t . This coordinate is not known to Blue (who conducts the strike), nor is it known to Gray (the analyst). However, Blue and Gray have a probability $dP(x, dx)$ that the middle point lies in the interval from $x-1/2dx$ through $x+1/2dx$. This probability has the form

$$dP(x, dx) = g(x)dx \quad (5-1)$$

Here $g(x)$ is the "position distribution function" (PDF). Since the middle point coordinate is a function of t , so is the PDF.

The PDF must reflect two sources of uncertainty. The first is the uncertainty associated with the reconnaissance information that, at reconnaissance time $t_r = 0$, the column's middle point occupied the position $x = 0$. This uncertainty is customarily referred to as "Target Location Error" (TLE). It is also customary to model the TLE by a Gaussian distribution function of the form

$$g_0(x_0) = \frac{1}{\sigma_0 \sqrt{2\pi}} \exp \left\{ -\frac{x_0^2}{2\sigma_0^2} \right\} \quad (5-2)$$

Here, x_0 is the actual position of the column's middle point, while

$$\bar{x}_0 = \langle x_0 \rangle = 0 \quad (5-3)$$

are the most probable and the mean positions, all referring to reconnaissance time $t_r = 0$.

Since the distribution function $g_0(x_0)$ is "induced" by the TLE committed by reconnaissance, we refer to it as "Reconnaissance Induced Position Distribution Function" (RPDF).

The second source of uncertainty is the uncertainty about the column's average velocity between reconnaissance and strike. This uncertainty is described by the "Velocity Induced Position Distribution Function" $g(x|x_0)$. This function is subject to the condition that the position at reconnaissance time $t_r = 0$ was x_0 .

The PDF, $g(x)$, occurring in equation (1) is the convolution of the RPDF, $g_0(x_0)$, and the VPDF, $g(x|x_0)$, that is,

$$g(x) = \int_{-\infty}^{+\infty} g(x|x_0) g_0(x_0) dx_0 \quad (5-4)$$

In the present section, we assume a symmetric VDF, $f(v)$. Naturally, the VPDF, $g(x|x_0)$, is then also symmetric. It is now our aim to derive this function.

To this end, we first establish the basic relations between the column's velocity and middle point position. We have

$$x = x(t) = x_0 + v t \quad (5-5)$$

Now, with respect to $g(x|x_0)$, x_0 is "given", that is, constant. Therefore, on account of equation (5), the "velocity uncertainty" dv (not to be confused with the differential), induces the "position uncertainty"

$$dx = t dv \quad (5-6)$$

It also follows under observance of relations (3) that

$$\bar{x}(t) = \langle x(t) \rangle = t \langle v \rangle \quad (5-7)$$

where $\bar{x}(t)$ and $\langle x(t) \rangle$ are the most probable and the mean value of the column's middle point coordinate at strike time t .

The linearity of relations (5), (6), and (7) implies that there is a biunique (reversibly unique) correspondence between the set of all velocity values and the set of all position coordinates, given x_0 . Therefore, the probability of a given velocity equals the probability of the corresponding position. Thence we have

$$dP \{ x, dx | x_0 \} = dP \{ v, dv \} \quad (5-8)$$

Using here the VDF and the VPDF, this reads

$$g(x|x_0) dx = f(v) dv \quad (5-9)$$

If here we substitute v and dv from equations (5) and (6), we obtain after cancellation of dx

$$g(x|x_0) = \frac{1}{t} f\left(\frac{x-x_0}{t}\right) \quad (5-10)$$

And if we observe here equations (2-1) and (7), we obtain

$$g(x|x_0) = \frac{1}{\sigma_{VP} \sqrt{2\pi}} \exp \left\{ -\frac{(x-x_0-\bar{x}(t))^2}{2\sigma_{VP}^2} \right\} \quad (5-11)$$

where

$$\sigma_{VP} = t \sigma_V \quad (5-12)$$

is the standard deviation of the VPDF.

For a brief discussion of results (11) and (12), we consider figure 2. It shows two trends:

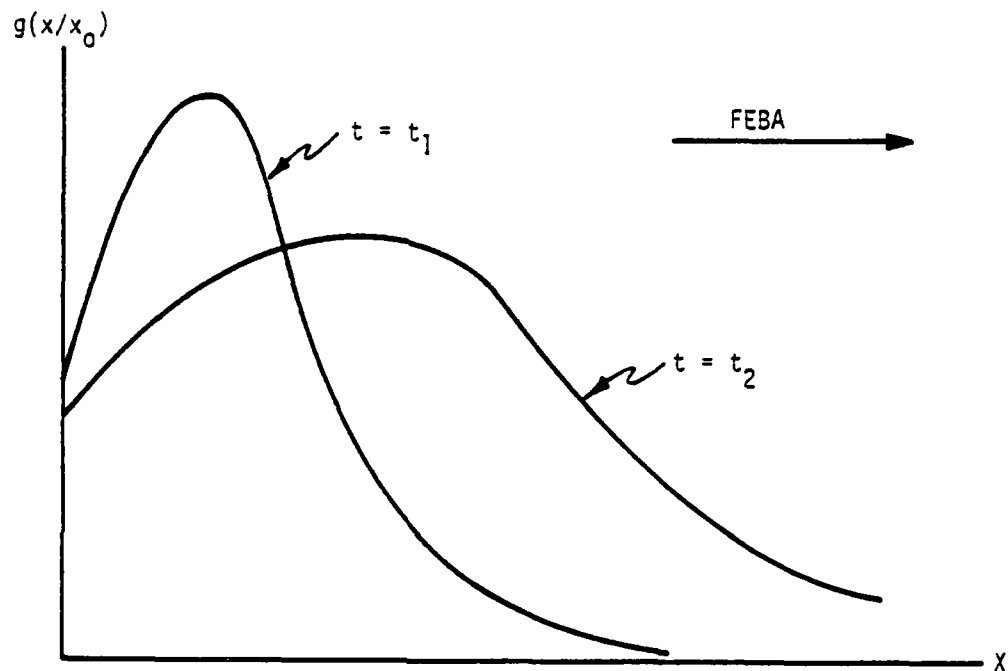


Figure 2. Velocity Induced Position Distribution Functions,
 $t_1 < t_2$, $\bar{v} > 0$

1. With increasing strike time t , the VPDF travels to the right. This trend reflects the column's motion toward the FEBA.
2. With increasing strike time t , the maximum of the VPDF decreases, while its width increases. This trend reflects the increase of the position uncertainty with increasing time.

Returning now to equation (4), we observe that both $g_0(x_0)$ and $g(x|x_0)$ are Gaussian. Therefore, the convolution is also Gaussian, and the result is

$$g(x) = \frac{1}{\sigma_p \sqrt{2\pi}} \text{Exp} \left\{ -\frac{((x-\bar{x}(t))^2)}{2\sigma_p^2} \right\} \quad (5-13)$$

where

$$\sigma_p = \sqrt{\sigma_0^2 + \sigma_{vp}^2} \quad (5-14)$$

is the standard deviation of the PDV.

For a brief discussion of results (13) and (14), we observe that $g(x)$ is a function of the strike time t by virtue of the two time functions $\bar{x}(t)$ and $\sigma_{vp}(t)$. Therefore, $g(x)$ shows the same trends as $g(x|x_0)$, illustrated in figure 2. It also satisfies the normalization condition

$$\int_{-\infty}^{+\infty} g(x) dx = 1 \quad (5-15)$$

F. ASYMMETRIC POSITION DISTRIBUTION FUNCTION

If we assume the asymmetric VDF from section D, the equations become more complicated and, eventually, lead to two integrals which cannot be solved in closed form (except for $\sigma_0 = 0$). However, the reasoning presented in the previous section remains essentially unchanged.

We first note that equations (5-1) through (5-10) are unaffected by the special properties of the VDF, $f(v)$. However, equations (5-11) and (5-12) must now be replaced as follows.

$$g(x|x_0) = \begin{cases} \frac{\sigma_1}{\sigma_v} g_1(x|x_0) & x-x_0 \leq \bar{x}(t) \\ \frac{\sigma_2}{\sigma_v} g_2(x|x_0) & x-x_0 \geq \bar{x}(t) \end{cases} \quad (6-1)$$

$$g_k(x|x_0) = \frac{1}{\sigma_{kp} \sqrt{2\pi}} \exp \left\{ -\frac{(x-x_0-\bar{x}(t))^2}{2\sigma_{kp}^2} \right\} \quad (k = 1, 2) \quad (6-2)$$

$$\sigma_{kp} = t\sigma_k \quad (k = 1, 2) \quad (6-3)$$

If we then enter equation (5-4) with equation (1), we obtain

$$g(x) = \frac{\sigma_1}{\sigma_v} \int_{x-\bar{x}(t)}^{+\infty} g_1(x|x_0) g_0(x_0) dx_0 \\ + \frac{\sigma_2}{\sigma_v} \int_{-\infty}^{x-\bar{x}(t)} g_2(x|x_0) g_0(x_0) dx_0 \quad (6-4)$$

The two integrals from equation (4) must be solved either numerically or approximately. This subject will not be pursued in the present report. It should be noted, however, that for $\sigma_0 = 0$, $g_0(x_0)$ goes over into Dirac's Delta Function. In this case, of course, the integrals from equation (4) are easily solved, and we obtain

$$g(x) = \begin{cases} \frac{\sigma_1}{\sigma_v} g_1(x) & x \leq \bar{x}(t) \\ \frac{\sigma_2}{\sigma_v} g_2(x) & x \geq \bar{x}(t) \end{cases} \quad (6-5)$$

with

$$g_k(x) = \frac{1}{\sigma_{kp} \sqrt{2\pi}} \exp \left\{ - \frac{(x - \bar{x}(t))^2}{2\sigma_{kp}^2} \right\} \quad (k = 1, 2) \quad (6-6)$$

G. THE PROBABILITY OF OCCUPANCY

Let $P_0(T)$ denote the probability that, at strike time t , and before any bombing, the "contemplated target point" T is occupied by the tank column. The subscript zero stands for the qualifier "before any bombing". We shall later also consider the probabilities of occupancy after the first bombing, second bombing, and so forth. They will be denoted by the symbols $P_1(T)$, $P_2(T)$, and so forth.

Figure 3 shows the contemplated target point T with the coordinate x_t , and two possible middle point positions M_1 and M_2 with the coordinates $x_t - L$ and $x_t + L$, where L is the half-length of the tank column. The points M_1 and M_2 are the lower and upper limits between which the middle point must lie if the column is to contribute to the occupancy of T . Therefore,

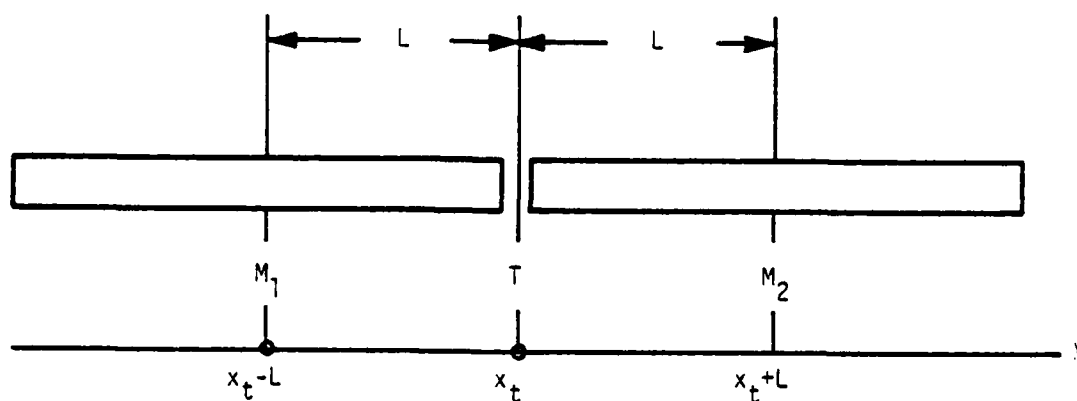


Figure 3. Limits of Middle Point Positions Contributing to Occupancy of T

the probability of occupancy, $P_0(T)$, is the probability that the column's middle point lies in the interval from x_t-L to x_t+L . The latter probability is obtained by integrating the PDF, $g(x)$, over the aforementioned interval. This yields

$$P_0(T) = \int_{x_t-L}^{x_t+L} g(x) dx \quad (7-1)$$

Instead of a copious discussion of the function $P_0(T)$, we offer figure 4. Although $P_0(T)$ is not a Gaussian distribution function, but a probability, it has the characteristic bell-shape. Its dependency on L and t is illustrated by considering two values of each.

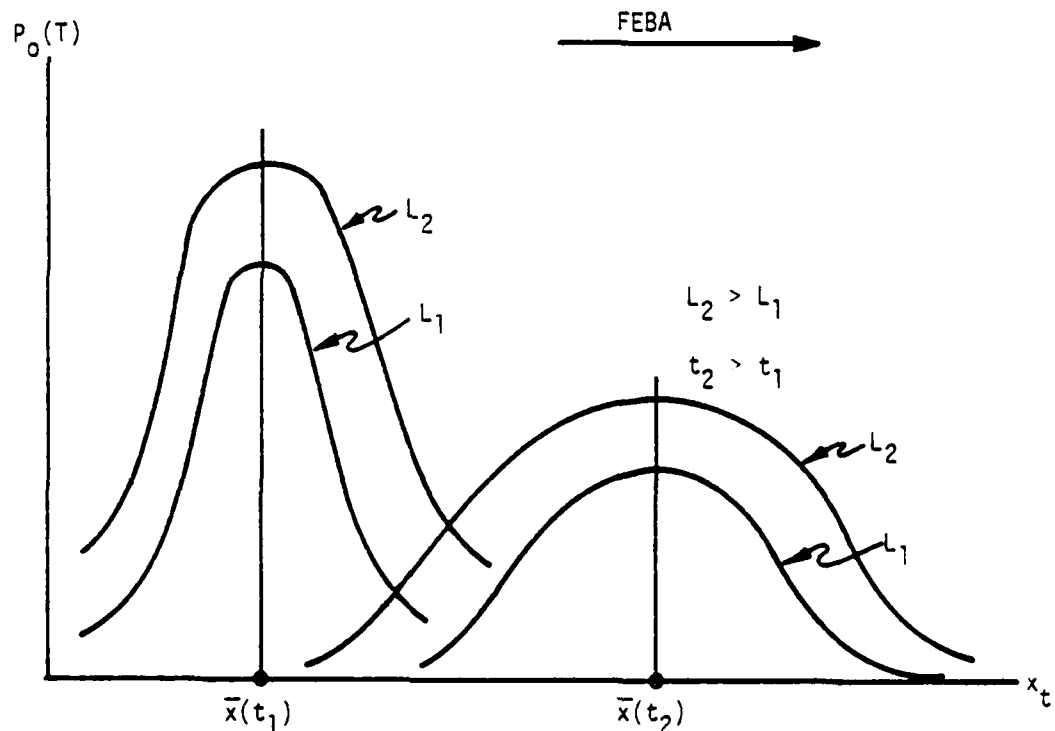


Figure 4. Probability of Occupancy, $\bar{v} > 0$

H. THE EXTERIOR TANK DENSITY

We assume that the tanks are continuously and homogeneously¹ distributed within the column. If the column has the length $2L$ and contains N tanks, the number of tanks per unit length is

$$n = \frac{N}{2L} \quad (3-1)$$

If then we denote the tank density by ρ , we have

$$\rho = \begin{cases} n & \text{within} \\ 0 & \text{without} \end{cases} \quad (3-2)$$

However, Blue and Gray do not know the column's position at strike time t . Therefore, they do not know ρ as a function of x . But there is another tank density which is known to Blue and Gray as a function of x . For clarity and convenience, we refer to ρ as the "interior density", and to the density now to be introduced as the "exterior density".

We denote the exterior density by the symbol $v_0(x_t)$. Here the subscript zero has the same function as at the symbol $P_0(T)$, in other words, it is a reminder that $v_0(x_t)$ refers to the state of affairs before any bombing.

The exterior tank density is defined as follows. If dN denotes the number of tanks within the interval from $x_t - 1/2dx_t$ to $x_t + 1/2dx_t$, then

$$dN = v_0(x_t) dx_t \quad (3-3)$$

1. It is possible to assume a heterogeneous tank distribution, that is, a "structured" column. While this does not pose any particular difficulties, it lengthens the report and is therefore disregarded.

Since the total number of tanks is N , it follows that

$$\int_{-\infty}^{+\infty} v_0(x_t) dx_t = N \quad (8-4)$$

Obviously, the exterior tank density must be related to the probability of occupancy. Indeed, if the contemplated target point T is occupied, then $v_0(x_t) = n$. But the probability that T is occupied is $P_0(T)$. Therefore,

$$v_0(x_t) = nP_0(T) \quad (8-5)$$

If equation (5) is substituted into equation (4) and if, at the same time, equation (1) is heeded, we obtain

$$\int_{-\infty}^{+\infty} P_0(T) dx_t = 2L \quad (8-6)$$

And if then equation (6-1) is observed, it follows that

$$\int_{-\infty}^{+\infty} \left\{ \int_{x_t-L}^{x_t+L} g(x) dx \right\} dx_t = 2L \quad (8-7)$$

We now assert that equation (7) is an identity, to wit, that it is true for all values of L . To prove the assertion, we introduce the special symbol

$$\Gamma(L) = \int_{-\infty}^{+\infty} \left\{ \int_{x_t-L}^{x_t+L} g(x) dx \right\} dx_t \quad (8-8)$$

The assertion is proved if we have demonstrated that $\Gamma(L) = 2L$.

The proof makes use of a well-known theorem of the calculus. Let the function $F(y)$ be defined as follows:

$$F(y) = \int_{\psi_1(y)}^{\psi_2(y)} f(x,y) dx \quad (8-9)$$

Here, $f(x,y)$, $\psi_1(y)$, and $\psi_2(y)$ may be any function's subject only to certain conditions of continuity and differentiability. The aforementioned theorem then assumes the form

$$\frac{dF}{dy} = \int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial f}{\partial y} dx + \frac{d\psi_2}{dy} f(\psi_2, y) - \frac{d\psi_1}{dy} f(\psi_1, y) \quad (8-10)$$

We apply this theorem to equation (8) for the purpose of forming the derivative $d\Gamma/dL$. Since equation (8) is a "repeated integral", the theorem must be applied twice. The first application yields

$$\frac{d\Gamma}{dL} = \int_{-\infty}^{+\infty} \left\{ \frac{d}{dL} \int_{x_t-L}^{x_t+L} g(x) dx \right\} dx_t \quad (8-11)$$

The second application yields

$$\frac{d\Gamma}{dL} = \int_{-\infty}^{+\infty} \left\{ g(x_t + L) + g(x_t - L) \right\} dx_t \quad (8-12)$$

There the substitutions $x_t + L = \xi$

and $x_t - L = \eta$ yield

$$\frac{d\Gamma}{dL} = \int_{-\infty}^{+\infty} g(\xi) d\xi + \int_{-\infty}^{+\infty} g(\eta) d\eta \quad (8-13)$$

Now we observe the normalization condition (5-15). Equation (13) then assumes the form

$$\frac{d\Gamma}{dL} = 2 \quad (8-14)$$

It follows by integration that

$$\Gamma(L) = 2L + \text{const} \quad (8-15)$$

If here we set $L = 0$, we get

$$\Gamma(0) = \text{const} \quad (8-16)$$

Hence we have

$$\Gamma(L) = 2L + \Gamma(0) \quad (8-17)$$

But from equation (8) can be seen that $\Gamma(0)=0$. Hence, it follows that

$$\Gamma(L) = 2L \quad (8-18)$$

which was to be proved.

I. ATTACK GEOMETRY

Figure 5 illustrates the attack geometry. It shows the two Cartesian coordinate systems (x, y) and (ξ, η) with all axes parallel to the ground. The three points A, I, T and their coordinates are defined in table 1.

Table 1

DEFINITION OF POINTS AND COORDINATES

NAME	SYMBOL	COORDINATES	RADIUS VECTOR
Aimpoint	A	x_a, y_a, ξ_a, η_a	\bar{r}_a
Impact Point	I	x_i, y_i, ξ_i, η_i	\bar{r}_i
Target Point	T	x_t, y_t, ξ_t, η_t	\bar{r}_t

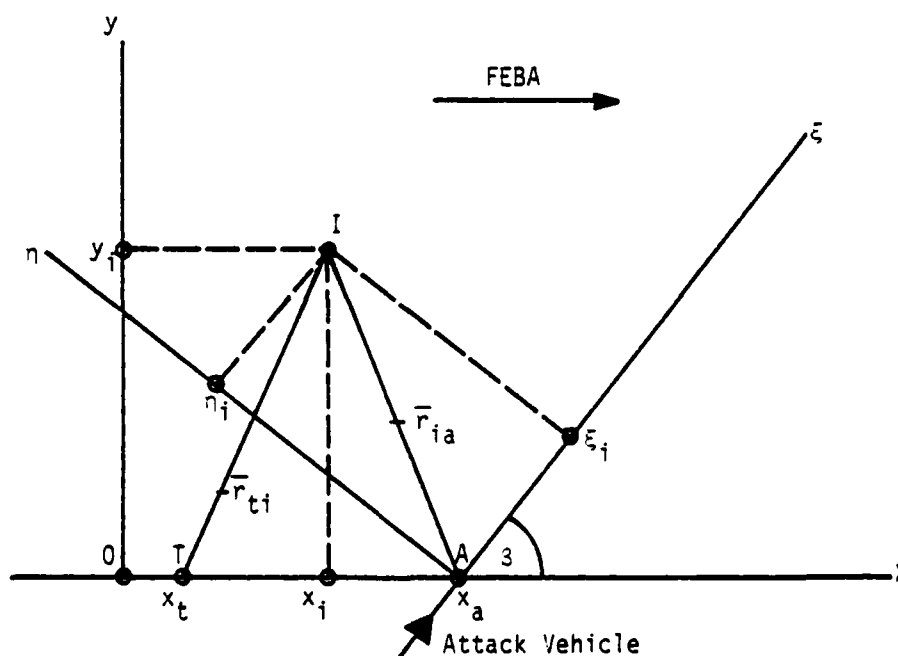


Figure 5. Attack Geometry

As before, the road on which the tank column moves toward the FEBA serves as x-axis. The y-axis serves to measure the offset from the road. The trajectory of the attack vehicle (aircraft or missile), serves as ξ -axis. The η -axis measures the offset from the trajectory of the attacker. The angle between x-axis and ξ -axis is denoted by β and called the "attack angle".

The contemplated target point, T, may lie anywhere on the road. The reason is that, with every point on the road (except the infinities), a non-vanishing probability is associated that it may be occupied by the column. Therefore, every point on the road is a "potential" target point. However, since the tanks are confined to the road, it follows that

$$y_t = 0 \quad (9-1)$$

Since the potential target points, T, are confined to the road, so are the aimpoints, A, that is,

$$y_a = 0 \quad (9-2)$$

The impact point, I, is randomly related to the aimpoint, A. Thence, it has a generally non-vanishing offset, y_i , from the road.

The radius vectors \bar{r}_a , \bar{r}_i , and \bar{r}_t , shown in table 1, but not in figure 5, refer to the origin, 0, of the (x, y) coordinate system. Shown in figure 5 are the "distance vectors"

$$\bar{r}_{ti} = \bar{r}_i - \bar{r}_t \quad (9-3a)$$

$$\bar{r}_{ta} = \bar{r}_a - \bar{r}_t \quad (9-3b)$$

$$\bar{r}_{ia} = \bar{r}_a - \bar{r}_i \quad (9-3c)$$

They point, in this order, from T to I, from T to A, and from I to A. For the norms of these vectors, we obtain under observance of equations (1) and (2) the relations

$$r_{ti}^2 = (x_i - x_t)^2 + y_i^2 \quad (9-4a)$$

$$r_{ta}^2 = (x_a - x_t)^2 + y_t^2 \quad (9-4b)$$

$$r_{ia}^2 = (x_a - x_i)^2 + y_i^2 \quad (9-4c)$$

From figure 5, we read the following equations of coordinate transformation:

$$x = x_a + \xi \cos \beta - \eta \sin \beta \quad (9-5a)$$

$$y = \xi \sin \beta + \eta \cos \beta \quad (9-5b)$$

$$\xi = (x - x_a) \cos \beta + y \sin \beta \quad (9-6a)$$

$$\eta = -(x - x_a) \sin \beta + y \cos \beta \quad (9-6b)$$

J. THE AIMPOINT RELATED KILL PROBABILITY

The ordinary kill probability is "impact point related". It has the form

$$P_k(T|I) = f(r_{ti}) \quad (10-1)$$

This is the probability that a target at T is killed by a munition impacting at I. Generally, this probability is a function of the distance r_{ti} from T to I, as indicated. In the present report, we consider the impact point related kill probability, that is, the function $f(r_{ti})$, as given by sources external to this report.

The object of the present report is the "aimpoint related kill probability" $P_k(T|A)$. This is the probability that a target at T is killed by a munition aimed at A . For the calculation of this probability, we need the probability that a munition aimed at A will impact at I . Denoting this probability by $dP(I|A)$, we have

$$P_k(T|A) = \int_I P_k(T|I) dP(I|A) \quad (10-2)$$

Here, the integration is to be extended over all possible impact points, I .

As usual, we generate the probability $dP(I|A)$ by an "Impact Distribution Function" (IDF). Denoting the IDF by the symbol $h(I|A)$, we have

$$dP(I|A) = h(I|A) dx_i dy_i \quad (10-3)$$

Substituting equation (3) into equation (2), we obtain

$$P_k(T|A) = \iint_{-\infty}^{+\infty} P_k(T|I) h(I|A) dx_i dy_i \quad (10-4)$$

If here the functions $P_k(T|I)$ and $h(I|A)$ are known, the aimpoint related kill probability $P_k(T|A)$ can be calculated on the basis of this equation.

Strictly speaking, the IDF lies outside of the scope of this report. However, since it is convenient and simple, we may as well provide this function at this occasion. We first note from figure 5 that the trajectory of the attacker has ξ -direction. Hence, we use for the formulation of the IDF the (ξ, η) -coordinate system, switching to the (x, y) -coordinate system thereafter. Also noting from figure 5 that the aimpoint, A , has the coordinates

Here, $P_0(T)$ and $P_k(T|A)$ are given by equations (7-1) and (10-4). The corresponding exterior tank density is

$$v_1(x_t) = nP_1(T) \quad (11-2)$$

This is the tank density at the arbitrary point T after one bomb has been aimed at A_1 .

After m bombs have been aimed at A_1, \dots, A_m , the probability of occupancy of T is

$$P_m(T) = P_0(T) \prod_{\alpha=1}^m \left\{ 1 - P_k(T|A_\alpha) \right\} \quad (11-3)$$

and the corresponding exterior tank density is

$$v_m(x_t) = nP_m(T) \quad (11-4)$$

Let then $\langle N_s | A_1, \dots, A_m \rangle$ and $\langle N_k | A_1, \dots, A_m \rangle$, respectively, denote the expected numbers of tanks that have survived and that have been killed after m bombs have been aimed at A_1, \dots, A_m . If it is then recalled that the total number of tanks in the column is N , it follows that

$$\langle N_s | A_1, \dots, A_m \rangle + \langle N_k | A_1, \dots, A_m \rangle = N \quad (11-5)$$

On the other hand, we also have

$$\langle N_s | A_1, \dots, A_m \rangle = \int_{-\infty}^{+\infty} v_m(x_t) dx_t \quad (11-6)$$

Hence, it follows that

$$\langle N_k | A_1, \dots, A_m \rangle = N - n \int_{-\infty}^{+\infty} P_m(T) dx_t \quad (11-7)$$

If here equation (3) is observed, we obtain

$$\langle N_k | A_1, \dots, A_m \rangle = N - n \int_{-\infty}^{+\infty} P_0(T) \prod_{\alpha=1}^m \left\{ 1 - P_k(T|A_\alpha) \right\} dx_t \quad (11-8)$$

$$\xi_a = 0 \quad (10-5a)$$

$$\eta_a = 0 \quad (10-5b)$$

we write

$$h(I|A) = \frac{1}{2\pi\sigma_R\sigma_D} \text{Exp} \left\{ -\frac{\xi_i^2}{2\sigma_R^2} - \frac{\eta_i^2}{2\sigma_D^2} \right\} \quad (10-6)$$

Here σ_R and σ_D are the standard deviations in "range" and "deflection", that is, in ξ -direction and in η -direction. The corresponding "errors probable" are related to the standard deviations by the equations

$$\text{REP} = 0.674\sigma_R \quad (10-7a)$$

$$\text{DEP} = 0.674\sigma_D \quad (10-7b)$$

For the transformation from the (ξ, η) -coordinates to the (x, y) -coordinates, we infer from equations (9-6a) and (9-6b) that

$$\xi_i = (x_i - x_a) \cos \beta + y_i \sin \beta \quad (10-8a)$$

$$\eta_i = -(x_i - x_a) \sin \beta + y_i \cos \beta \quad (10-8b)$$

If these equations are substituted into equation (6), the IDF is given in the form in which it is needed in equation (4).

K. EXPECTED NUMBER OF TANKS KILLED

We recall from section G that $P_0(T)$ denotes the probability of occupation of the contemplated target point T before any bombing. Let now $P_1(T)$ denote the probability of occupation of T after one bomb has been aimed at A . We then have

$$P_1(T) = P_0(T) \left\{ 1 - P_k(T|A) \right\} \quad (11-1)$$

For computational purposes, it is convenient to convert the product under the integral into sums of products, applying the formula

$$\prod_{i=1}^m (1-x_i) = 1 - \sum_{i=1}^m x_i + \sum_{i=1}^{m-1} \sum_{k=i+1}^m x_i x_k - \quad (11-9)$$

$$\sum_{i=1}^{m-2} \sum_{k=i+1}^{m-1} \sum_{l=k+1}^m x_i x_k x_l + - \dots$$

When applying this formula to equation (8), we observe in conjunction with equations (8-1) and (8-6) that

$$N = n \int_{-\infty}^{+\infty} P_0(T) dx_t \approx 0 \quad (11-10)$$

Equation (8) then assumes the form

$$\begin{aligned} \langle N_k | A_1, \dots, A_m \rangle &= n \int_{-\infty}^{+\infty} P_0(t) \sum_{i=1}^m P_k(T | A_i) dx_t \\ &= n \int_{-\infty}^{+\infty} P_0(T) \sum_{i=1}^{m-1} \sum_{k=i+1}^m P_k(T | A_i) P_k(T | A_k) dx_t + - + \dots \end{aligned} \quad (11-11)$$

Equation (11) is the general result for arbitrarily selected aimpoints A_1, \dots, A_m . For a brief discussion of this result, we consider the case $m = 1$. Equation (11) then reduces to

$$\langle N_k | A \rangle = n \int_{-\infty}^{+\infty} P_0(T) P_k(T | A) dx_t \quad (11-12)$$

If here equations (7-1), (8-1), and (10-4) are heeded, this becomes

$$\langle N_k | A \rangle = \frac{N}{2L} \int_{-\infty}^{+\infty} \left\{ \int_{x_t-L}^{x_t+L} g(x) dx \right\} \left\{ \int_{-\infty}^{+\infty} P_k(T|I) h(I|A) dx_i dy_i \right\} dx_t \quad (11-13)$$

Here we offer the following observations:

1. The aimpoint, A , is freely selectable. However, it is not difficult to see that, for a symmetric PDF, $g(x)$, the optimal aimpoint is the most probable position of the column's middle point, that is,

$$x_a = \bar{x}(t) \quad (11-14)$$

Since $\bar{x}(t) = t\bar{v}$, it follows that the optimal aimpoint travels with the column's most probable velocity toward the FEBA.

The optimal aimpoint for the asymmetric PDF will be determined in section L.

2. By virtue of the PDF, $g(x)$, the expectation value $\langle N_k | A \rangle$ depends on the strike time, t . More specifically, $\langle N_k | A \rangle$ decreases as t increases.

3. Also by virtue of $g(x)$, $\langle N_k | A \rangle$ depends on σ_0 and σ_v , the standard deviations associated with the TLE and the VDF. More specifically, $\langle N_k | A \rangle$ decreases as σ_0 and σ_v increase.

4. From equation (13) can be seen that $\langle N_k | A \rangle$ depends on the column's half-length, L . It can be shown that $\langle N_k | A \rangle$ decreases as L increases, provided N is kept constant.

5. By virtue of the impact point related kill probability, $P_k(T|I)$, $\langle N_k | A \rangle$ depends on the kill-potential and the kill-range of the munition.

6. By virtue of the IDF, $h(I|A)$, $\langle N_k|A \rangle$ depends:

- a. On the aimpoint selection.
- b. On the attack angle, β .
- c. On the delivery accuracy measured by σ_R and σ_D . $\langle N_k|A \rangle$ decreases as σ_R and σ_D increase.

L. OPTIMAL AIMPOINT FOR ASYMMETRIC POSITION DISTRIBUTION FUNCTION

We consider the case $m = 1$, that is, one bomb per tank column. In case of the symmetric PDF, it is obvious that the optimal aimpoint is identical to the most probable middle point position, $\bar{x}(t)$, which in turn, is identical to the mean middle point position, $\langle \bar{x}(t) \rangle$. However, in case of the asymmetric PDF (3-3), the mean middle point position lies somewhat to the right of the most probable position. For similar reasons, it is to be expected that the optimal aimpoint, too, lies somewhat to the right of $\bar{x}(t)$. This is illustrated in figure 6, where A^* denotes the optimal aimpoint with the coordinate x_a^* .

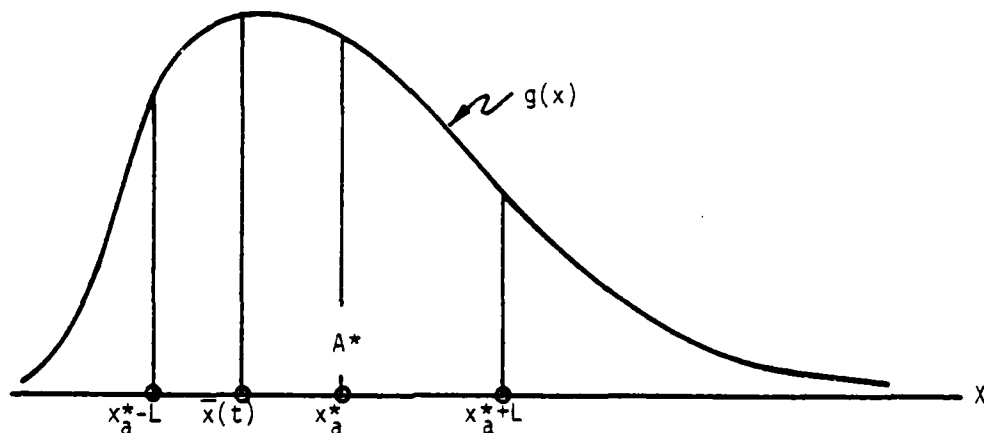


Figure 6. Optimal Aimpoint, A^*

Obviously, the condition for aimpoint optimization is

$$\frac{\partial p_0}{\partial x_t} = 0 \quad (12-1)$$

This equation must have a unique solution for x_t which we denote by x_a^* . We then recall from section G that $P_0(x_t)$ is generated by integration of $g(x)$ over the interval from $x_t - L$ to $x_t + L$. Since $g(x)$ is now asymmetric with respect to $\bar{x}(t)$, the question arises how $\bar{x}(t)$ relates to the interval of integration. In figure 6, we have tacitly assumed that $\bar{x}(t)$ lies within the interval, that is,

$$\text{Case I: } x_t - L \leq \bar{x}(t) \leq x_t + L \quad (12-2)$$

Principally, however, we also have

$$\text{Case II: } \bar{x}(t) \leq x_t - L$$

$$\text{Case III: } \bar{x}(t) \geq x_t + L$$

One can show, however, that Cases II and III do not yield a maximum of $P_0(x_t)$. This can also be inferred intuitively from figure 6. In order to save time and space, we proceed immediately to Case I.

If condition (2) is satisfied, it follows from equations (7-1), (3-1a), and (3-1b) that

$$P_0(x_t) = \frac{\sigma_1}{\sigma_v} \int_{x_t - L}^{\bar{x}(t)} g_1(x) dx + \frac{\sigma_2}{\sigma_v} \int_{\bar{x}(t)}^{x_t + L} g_2(x) dx \quad (12-3)$$

To form the derivative $\partial P_0 / \partial x_t$, we utilize the auxiliary theorem from section H. This yields

$$\frac{\partial P_0}{\partial x_t} = - \frac{\sigma_1}{\sigma_2} g_1(x_t - L) \frac{\partial}{\partial x_t} (x_t - L) - \frac{\sigma_2}{\sigma_v} g_2(x_t + L) \frac{\partial}{\partial x_t} (x_t + L) \quad (12-4)$$

If here we set $\partial P_0 / \partial x_t = 0$, we must replace x_t by x_a^* . This yields

$$\frac{\sigma_1}{\sigma_v} g_1(x_a^* - L) = \frac{\sigma_2}{\sigma_v} g_2(x_a^* + L) \quad (12-5)$$

Here we observe equation (6-6) and obtain

$$\frac{(x_a^* - L - \bar{x}(t))^2}{\sigma_1^2} = \frac{(x_a^* + L - \bar{x}(t))^2}{\sigma_2^2} \quad (12-6)$$

Taking the square root, we must make sure that both expressions are positive in accordance with condition (2). This yields

$$\frac{x_a^* - L - \bar{x}(t)}{\sigma_1} = \frac{\bar{x}(t) - x_a^* - L}{\sigma_2} \quad (12-7)$$

This equation is linear in x_a^* and has the solution

$$x_a^* = \bar{x}(t) + L \frac{\sigma_2 - \sigma_1}{\sigma_2 + \sigma_1} \quad (12-8)$$

This is the optimal aimpoint coordinate.

For a brief discussion of this result, we first consider the symmetric case $\sigma_1 = \sigma_2$. It yields $x_a^* = \bar{x}(t)$, as expected. Next we consider the case $L = 0$. It also yields $x_a^* = \bar{x}(t)$, as should be expected. Finally, we recall condition (2). It yields $x_a^* > \bar{x}(t)$, as expected.

M. DISTRIBUTION FUNCTIONS FOR RELATIVE VELOCITY AND SPACING BETWEEN COLUMNS

In the next section, we shall consider an aggregate of more than one column. The prerequisites for this problem are the "Relative Velocity Distribution Function" (RVDF), and the "Spacing Distribution Function" (SDF), which will now be derived.

Let $dP \left\{ (v, dv); (v-v_r, dv_r) \right\}$ denote the joint probability that one column has a velocity in the interval from $v - 1/2dv$ to $v + 1/2dv$, and the other column has a velocity in the interval from $v-v_r - 1/2 dv_r$ to $v-v_r + 1/2dv_r$. We then have

$$dP \left\{ (v, dv); (v-v_r, dv_r) \right\} = f(v) dv f(v-v_r) dv_r \quad (13-1)$$

Equation (1) describes one case of an infinite set of cases in all of which the two columns have the relative velocity v_r . Let $dP(v_r, dv_r)$ denote the probability of the total set, that is, the probability that the relative velocity between the two columns lies in the interval from $v_r - 1/2dv_r$ to $v_r + 1/2dv_r$. Introducing the RVDF, $f_r(v_r)$, we write

$$dP(v_r, dv_r) = f_r(v_r) dv_r \quad (13-2)$$

From the definition of $dP(v_r, dv_r)$ follows that it is obtained by integration of equation (1) over all values of v from $-\infty$ to $+\infty$. This yields

$$dP(v_r, dv_r) = dv_r \int_{-\infty}^{+\infty} f(v) f(v-v_r) dv \quad (13-3)$$

Observing here equation (2), we obtain

$$f_r(v_r) = \int_{-\infty}^{+\infty} f(v) f(v-v_r) dv \quad (13-4)$$

In other words, the RVDF is the convolution of the VDF with itself. We may also say that it is the "autocorrelation function".

If $f(v)$ is Gaussian, so is $f_r(v_r)$. Assuming now the symmetric VDF from section B, we have

$$f_r(v_r) = \frac{1}{\sigma_r \sqrt{2\pi}} \text{Exp} \left\{ -\frac{v_r^2}{2\sigma_r^2} \right\} \quad (13-5)$$

with

$$\sigma_r = \sigma_v \sqrt{2} \quad (13-6)$$

For a brief discussion of this result, we first observe that the most probable and the mean value of the relative velocity are

$$\bar{v}_r = \langle v_r \rangle = 0 \quad (13-7)$$

as should be expected. Next we observe that $\sigma_r > \sigma_v$. This relation reflects the fact that two independent columns constitute a system of two statistical degrees of freedom. Naturally, such a system leaves more room for error than a system with only one degree of freedom.

Next we turn to the "Spacing Distribution Function" (SDF). Let s_0 and s denote one half of the distances between the middle points of two adjacent columns (half-spacing) at reconnaissance time $t_r = 0$ and at strike time t . We then have

$$s = s_0 + v_r t \quad (13-8)$$

and

$$ds = t dv_r \quad (13-9)$$

Let then $dP(s, ds)$ denote the probability that the spacing lies between $s - 1/2 ds$ and $s + 1/2 ds$. Denoting the SDF by $F(s)$, we write

$$dP(s, ds) = F(s) ds \quad (13-10)$$

We then infer from the linearity of equations (8) and (9) that

$$dP(s, ds) = dP(v_r, dv_r) \quad (13-11)$$

Hence we have

$$F(s) ds = f_r(v_r) dv_r \quad (13-12)$$

Here, we observe equations (8) and (9). This yields

$$F(s) = \frac{1}{t} f_r \left(\frac{s-s_0}{t} \right) \quad (13-13)$$

And this yields

$$F(s) = \frac{1}{\sigma_s \sqrt{2\pi}} \text{Exp} \left\{ - \frac{(s-s_0)^2}{2 \sigma_s^2} \right\} \quad (13-14)$$

with

$$\sigma_s = t \sigma_r \quad (13-15a)$$

or

$$\sigma_s = \sqrt{2} \quad t \sigma_v \quad (13-15b)$$

For a brief discussion of this result, we first observe that the most probable and the mean value of s are identical to the original reconnaissance estimate s_0 , that is,

$$\bar{s} = \langle s \rangle = s_0 \quad (13-16)$$

This is in agreement with relation (7). Next, we observe that σ_s increases with time. This simply reflects the general rule that a velocity error leads to a time-proportional position error.

N. AGGREGATES OF MORE THAN ONE COLUMN

If there are more than one column on the road, the probability of occupancy, $P_0(T)$, is increased because each column is a potential contributor to the occupancy of T . We now assume that reconnaissance has detected an "aggregate" of M columns on the road march toward the FEBA. For convenience, we assume that M is odd. In this case, the aggregate's middle point is the middle point of the middle column. At reconnaissance time $t_r = 0$, this middle point occupies the position $x = 0$.

Also for convenience, we assume that all columns are equal. This implies that they have the same half-length, L , that they contain the same number of tanks, N , and - most important - that they obey the same VDF, $f(v)$.

The challenge of the present problem lies in the task of providing a realistic description of the interdependency of the columns in analytical terms. Certainly, the columns may have statistically different velocities, though they obey the same VDF, but they must not penetrate each other or change their order. Hence, they are not quite independent of each other, but they are not rigidly connected, either.

For the analytical treatment of this problem, we derived in the previous section the SDF, $F(s)$. From the derivation of this function follows that it assumes complete independence of the columns. This can also be inferred from the fact that s is allowed to assume any value including negative values. This, however, violates the condition that the columns must not penetrate each other or change their order. This condition will now be enforced by "renormalization" of the SDF.

We recall that L denotes the half-length of the single column, and that s denotes the half-spacing, that is, half of the distance between column middle points. Using these two concepts, we now express the condition of no penetration by the following inequality:

$$s \geq L$$

(14-1)

This condition is satisfied if we replace the SDF, $F(s)$, by the following function

$$F_R(s) = \begin{cases} R(t) F(s) & s \geq L \\ 0 & s < L \end{cases} \quad (14-2)$$

Here, $R(t)$ is the "renormalization function" which, as indicated, depends on the time. It is the solution of the "renormalization condition"

$$R(t) \int_L^{+\infty} F(s) ds = 1 \quad (14-3a)$$

which, in view of equation (2), may also be written in the form

$$\int_{-\infty}^{+\infty} F_R(s) ds = 1 \quad (14-3b)$$

The function $F_R(s)$ is called "Renormalized Spacing Distribution Function" (RSDF). It satisfies the condition of no penetration while imposing no further conditions on the free mobility of the column.

Figure 7. Aggregate of Five Columns

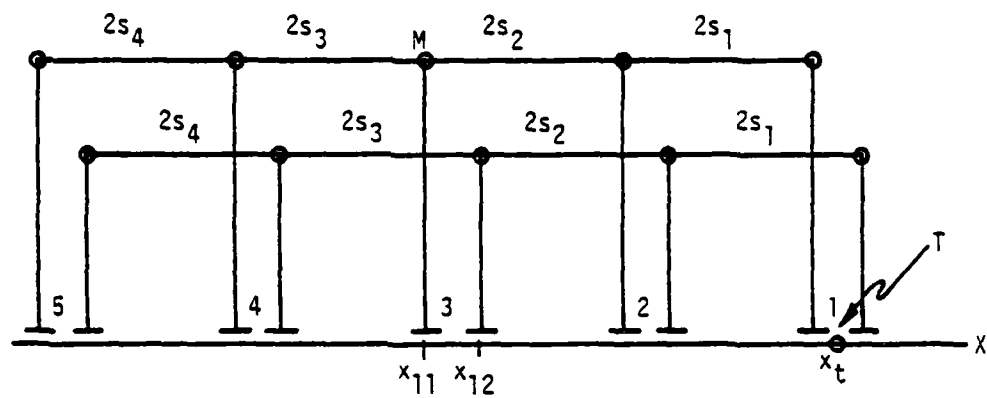


Figure 7a. Arrival and Departure of First Column

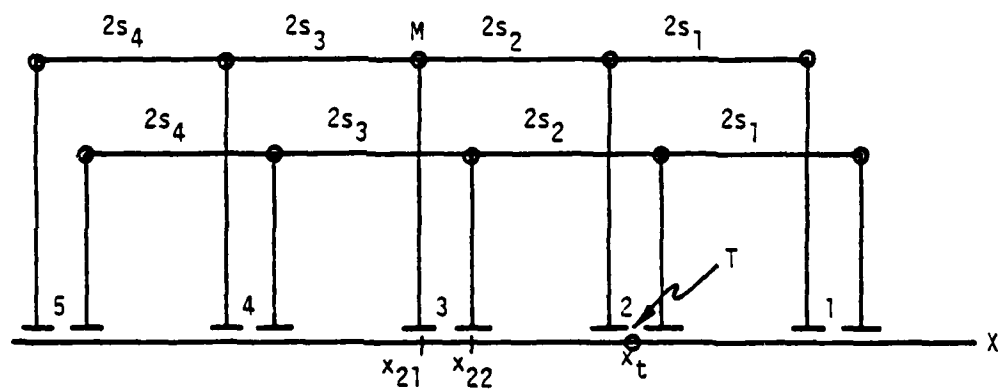


Figure 7b. Arrival and Departure of Second Column

Figures 7a and 7b show an aggregate of five columns, numbered 1 through 5 in the order in which they pass through any given point. The aggregate's middle point is denoted by M. In the present example, this is the middle point of the third column.

In both figures, the spacings between the middle points of adjacent columns are denoted by $2s_1$, $2s_2$, $2s_3$, and $2s_4$. These are independent random variables obeying the same RSDF.

In both figures, attention is focused on the contemplated target point T with the coordinate x_t . The upper and lower bars connect the five columns in different "situations". In figure 7a, the upper bar connects the columns in a situation in which the first column just "arrives" at T, while the lower bar depicts the situation where the first column "departs" from T. In figure 7b, the "arrival" and "departure" depicted by the upper and lower bars refer to the second column.

It is important to understand that the four situations depicted in figures 7a and 7b do not necessarily refer to four different times. At any given time each of the four situations is possible. Of course, the situations are mutually exclusive, that is, only one of them is possible at any given time.

To each of the four situations corresponds a position of the middle point of the aggregate. In figure 7a, the positions corresponding to the upper and lower bars are x_{11} and x_{12} , and in figure 7b, they are x_{21} and x_{22} . Hence the first index indicates the column, while the second index indicates arrival or departure.

We then read from figure 7a the following relations:

$$x_{11}(s_1, s_2) + 2(s_1 + s_2) - L = x_t \quad (14-4a)$$

$$x_{12}(s_1, s_2) + 2(s_1 + s_2) + L = x_t \quad (14-4b)$$

Here the notation $x_{11}(s_1, s_2)$ and $x_{12}(s_1, s_2)$ indicates that these positions are functions of s_1 and s_2 . For convenience, we combine the two equations (4a) and (4b) in the form

$$x_{1k}(s_1, s_2) + 2(s_1 + s_2) \pm L = x_t(k = 1, 2) \quad (14-5a)$$

where the upper and lower signs of L refer to the upper and lower bars of figure 7a or, what amounts to the same, to $k = 1$ and $k + 2$, indicating arrival and departure.

In a similar way, we obtain from figure 7b

$$x_{2k}(s_2) + 2s_2 \pm L = x_t(k = 1, 2) \quad (14-5b)$$

where the first index of x_{2k} indicates that now the second column is under consideration. If then we consider also the third, fourth, and fifth columns, we obtain

$$x_{3k} \pm L = x_t(k = 1, 2) \quad (14-5c)$$

$$x_{4k}(s_3) - 2s_3 \pm L = x_t(k = 1, 2) \quad (14-5d)$$

$$x_{5k}(s_3, s_4) - 2(s_3 + s_4) \pm L = x_t(k = 1, 2) \quad (14-5e)$$

The middle point positions $x_{1k}(s_1, s_2)$, $s_{2k}(s_2)$, x_{3k} , $x_{4k}(s_3)$, and $x_{5k}(s_3, s_4)$ are subject to the condition that the spacings assume the values listed in parenthesis. Therefore, if we integrate the PDF, $g(x)$, over the corresponding intervals, for example, from $x_{11}(s_1, s_2)$ to $x_{12}(s_1, s_2)$, we obtain the corresponding conditional probabilities of occupancy. We write

$$P_1(x_t | s_1, s_2) = \int_{x_{11}(s_1, s_2)}^{x_{12}(s_1, s_2)} g(x) dx \quad (14-6a)$$

$$P_2(x_t | s_2) = \int_{x_{21}(s_2)}^{x_{22}(s_2)} g(x) dx \quad (14-6b)$$

$$P_3(x_t) = \int_{x_{31}}^{x_{32}} g(x) dx \quad (14-6c)$$

$$P_4(x_t | s_3) = \int_{x_{41}(s_3)}^{x_{42}(s_3)} g(x) dx \quad (14-6d)$$

$$P_5(x_t | s_2, s_4) = \int_{x_{51}(s_3, s_4)}^{x_{52}(s_3, s_4)} g(x) dx \quad (14-6e)$$

We then proceed from the conditional to the unconditional probabilities of occupancy by convolution with the RSDF. This yields

$$P_1(x_t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1(x_t | s_1, s_2) F_R(s_1) F_R(s_2) ds_1 ds_2 \quad (14-7a)$$

$$P_2(x_t) = \int_{-\infty}^{+\infty} P_2(x_t | s_2) F_R(s_2) ds_2 \quad (14-7b)$$

$$P_3(x_t) = P_3(x_t) \quad (14-7c)$$

$$P_4(x_t) = \int_{-\infty}^{+\infty} P_4(x_t|s_3) F_R(s_3) ds_3 \quad (14-7d)$$

$$P_5(x_t) = \iint_{-\infty}^{+\infty} P_5(x_t|s_4, s_5) F_R(s_4) F_R(s_5) ds_4 ds_5 \quad (14-7e)$$

We now make certain simplifications which are significant in terms of the reduction of volume and labor, but rather insignificant in terms of the errors which they introduce. To that end, we observe from equations (5a) through (5e) that the s_i occur only as sums, for example, $s_1 + s_2$. Therefore, equation (7a), for example, has the form

$$P_1(x_t) = \iint_{-\infty}^{+\infty} \phi(s_1 + s_2) F_R(s_1) F_R(s_2) ds_1 ds_2 \quad (14-8)$$

We assert that this double integral can be approximated by the following single integral:

$$P_1(x_t) \approx \int_{-\infty}^{+\infty} \phi(2s) F_R(s) ds \quad (14-9)$$

Since the proof of the assertion is rather involved, it has been relegated to the appendix.

If these simplifications are carried out in equations (5a) through (5e), we obtain

$$x_{1k}(S) = x_t - 4S \mp L \quad (14-10a)$$

$$x_{2k}(S) = x_t - 2S \mp L \quad (14-10b)$$

$$x_{3k} = x_t \mp L \quad (k = 1, 2) \quad (14-10c)$$

$$x_{4k}(S) = x_t + 2S \mp L \quad (14-10d)$$

$$x_{5k}(S) = x_t + 4S \mp L \quad (14-10e)$$

Equations (6a) through (6e) then assume the following form

$$P_1(x_t | S) = \int_{x_{11}(S)}^{x_{12}(S)} g(x) dx \quad (14-11a)$$

$$P_2(x_t | S) = \int_{x_{21}(S)}^{x_{22}(S)} g(x) dx \quad (14-11b)$$

$$P_3(x_t | s) = \int_{x_{31}(S)}^{x_{12}(S)} g(x) dx \quad (14-11c)$$

$$P_4(x_t | S) = \int_{x_{41}(S)}^{x_{42}(S)} g(x) dx \quad (14-11d)$$

$$P_5 (X_t | S) = \int_{x_{51}(S)}^{x_{52}(S)} g(x) dx \quad (14-11e)$$

Equations (6a) through (6e) establish the conditioned probabilities that the first through fifth columns occupy T. Because of the condition of no penetration, these probabilities are mutually exclusive. Therefore, the probability that any one of the five columns occupies T is

$$P_0 (X_t | S) = \sum_{i=1}^5 P_i (X_t | S) \quad (14-12)$$

We then proceed from the conditional to the unconditional probability of occupancy by convolution with the RSDF. This yields

$$P_0 (X_t) = \int_{-\infty}^{+\infty} P_0 (X_t | S) F_R (S) ds \quad (14-13)$$

Equations (10), (11), (12), and (13) permit the calculation of the joint probability of occupancy of any point T on the road at any time by the columns of an aggregate of five. Extension to aggregates of M columns is straightforward.

We now consider a numerical example. It is assumed that, at reconnaissance time $t_r = 0$, the aggregate's middle point occupied the position $x = 0$. The fixed numerical assumptions are:

L	=	230 m
S_0	=	400 m
$\bar{v} = \langle v \rangle$	=	25 km/h
σ_v	=	10 km/h
σ_0	=	100 m

The numerical results are presented in figures (8a), (8b), and (8c). Shown on the abscissa and on the ordinate are the strike time t (delay between reconnaissance and strike) and the probability of occupancy, $P_o(x_t)$. The number of columns, M , is odd and parameterized from 1 through 7. The three graphs refer, in this order, to the coordinates $x_t = 3$ km, $x_t = 4$ km, and $x_t = 5$ km.

We make the following observations:

1. The probability of occupancy shows the typical decline with increasing time. This effect would also show up in the expected number of tanks killed, if it were calculated.
2. The probability of occupancy increases, in certain cases, rather strongly, as the number of columns increases. This effect tends to soften the decline with increasing time.
3. An increase of x_t has about the same effect as an increase of t .

Figure 8. Aggregate of M Columns

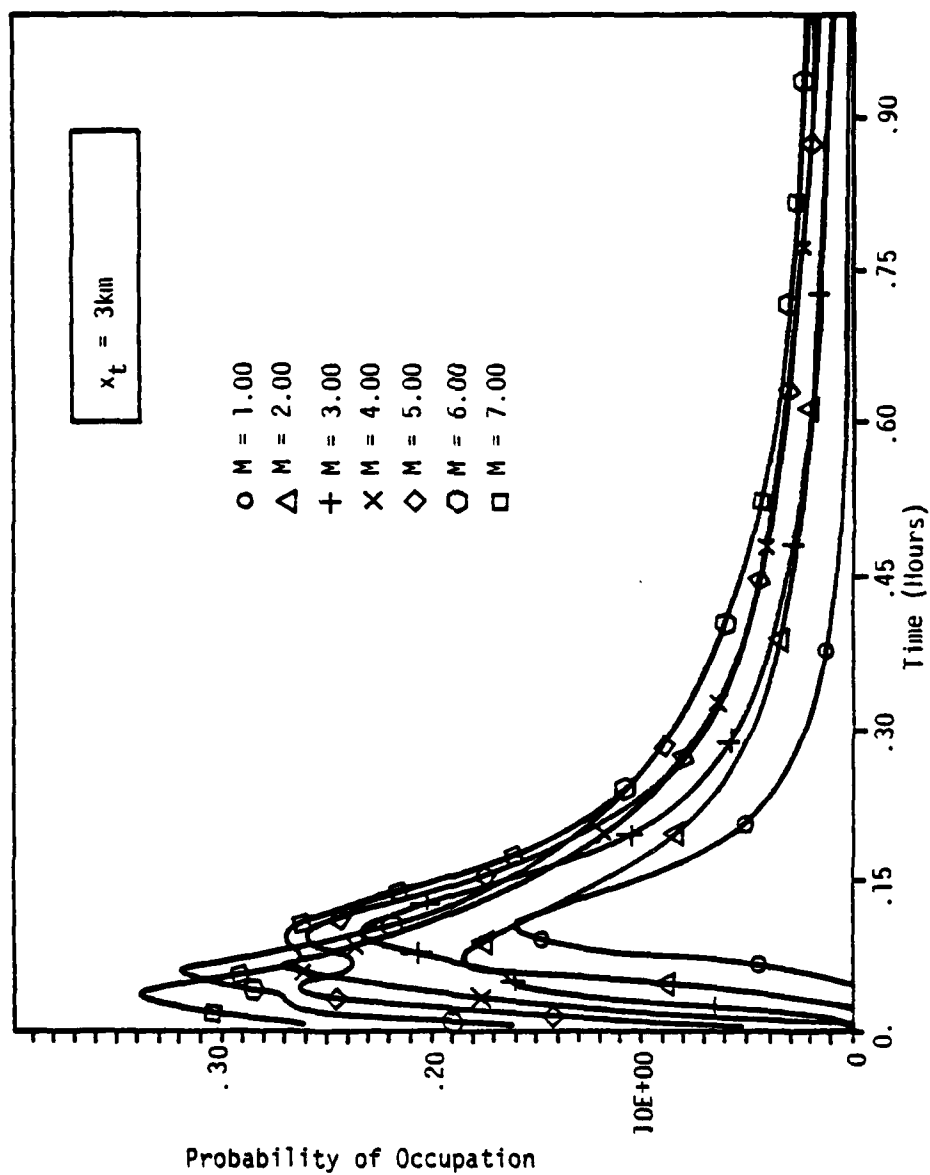


Figure 8a. Aggregate of M Columns, $x_t = 3 \text{ km}$

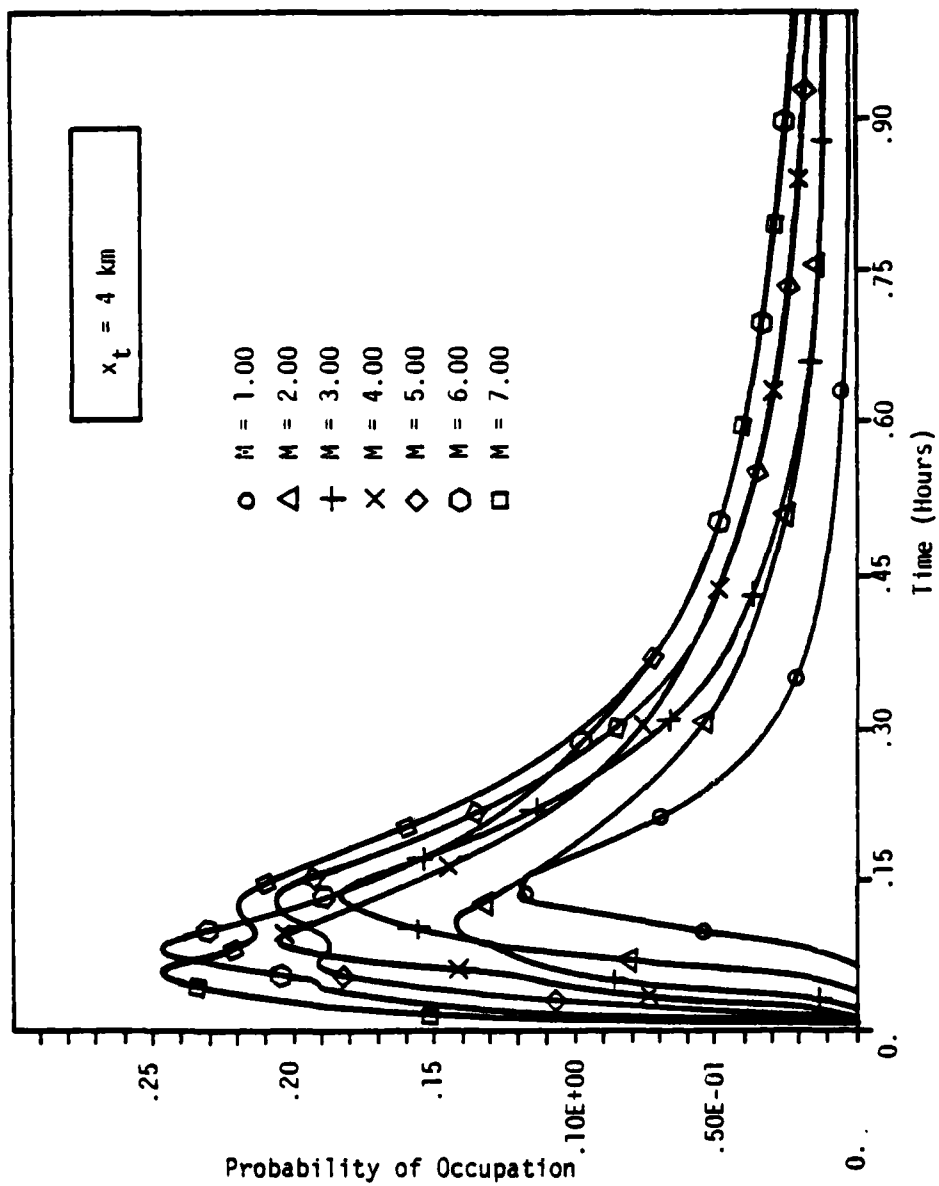


Figure 8b. Aggregate of M Columns, $x_t = 4 \text{ km}$

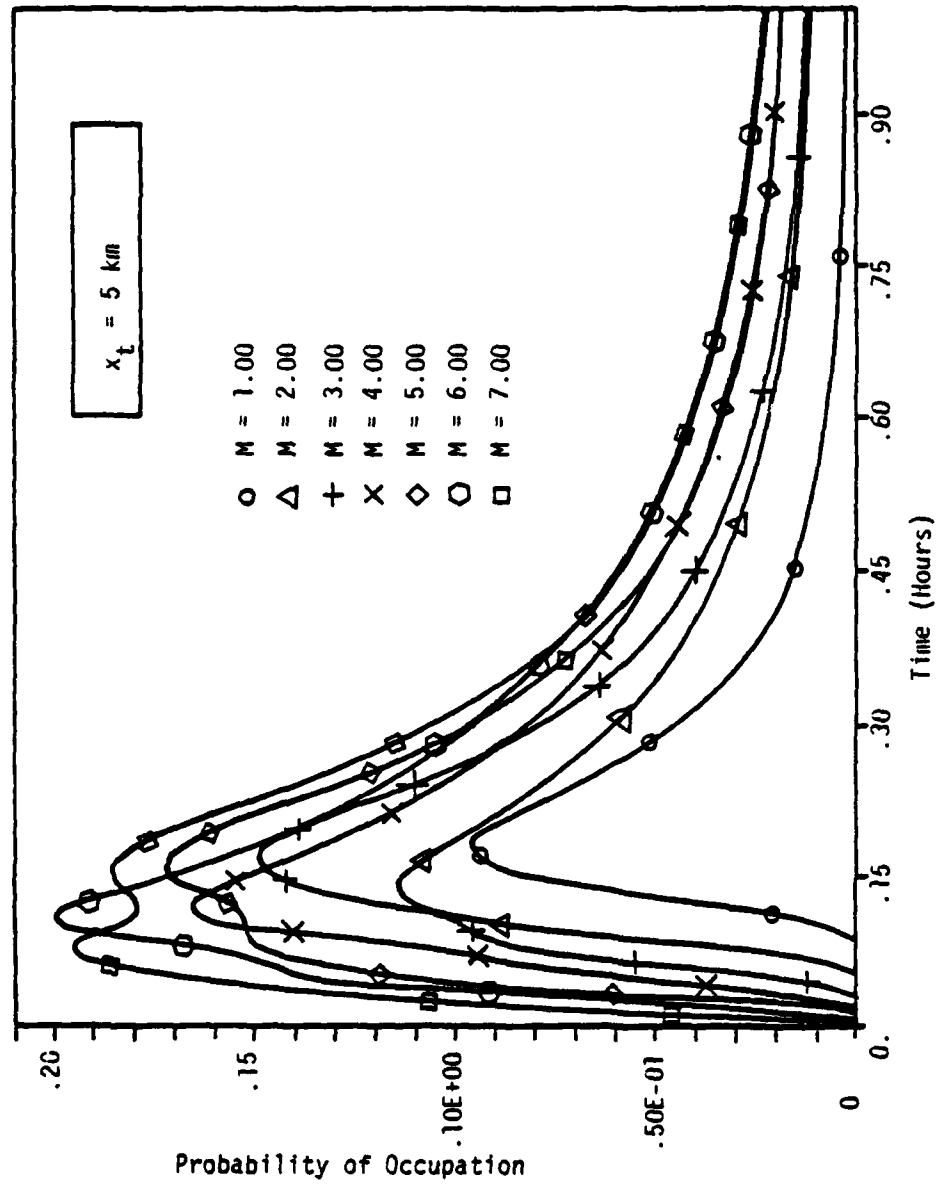


Figure 8c. Aggregate of M Columns, $x_t = 5 \text{ km}$

0. THE TWO-DIMENSIONAL TANK ARRAY

By "two-dimensional array", we mean two things. First, the tank array has two dimensions. Second, it moves in two dimensions. We employ a Cartesian coordinate system (x, y) with both axes parallel to the ground and the x-axis pointing toward the FEBA (see figure 9). We assume that the tank array has a rectangular shape and that its sides remain at all times parallel to the coordinate axes. At reconnaissance time, $t_r = 0$, the middle point shall occupy the position $x = 0, y = 0$.

With regard to the motion of the array, we assume that the velocities v_x and v_y obey the independent, Gaussian VDFs

$$f_x(v_x) = \frac{1}{\sigma_{vx} \sqrt{2\pi}} \exp \left\{ -\frac{(v_x - \bar{v}_x)^2}{2\sigma_{vx}^2} \right\} \quad (15-1a)$$

$$f_y(v_y) = \frac{1}{\sigma_{vy} \sqrt{2\pi}} \exp \left\{ -\frac{(v_y - \bar{v}_y)^2}{2\sigma_{vy}^2} \right\} \quad (15-1b)$$

Only in x-direction shall the array have a systematic velocity, that is

$$\bar{v}_x = \langle v_x \rangle > 0 \quad (15-2a)$$

In y-direction, we assume

$$\bar{v}_y = \langle v_y \rangle = 0 \quad (15-2b)$$

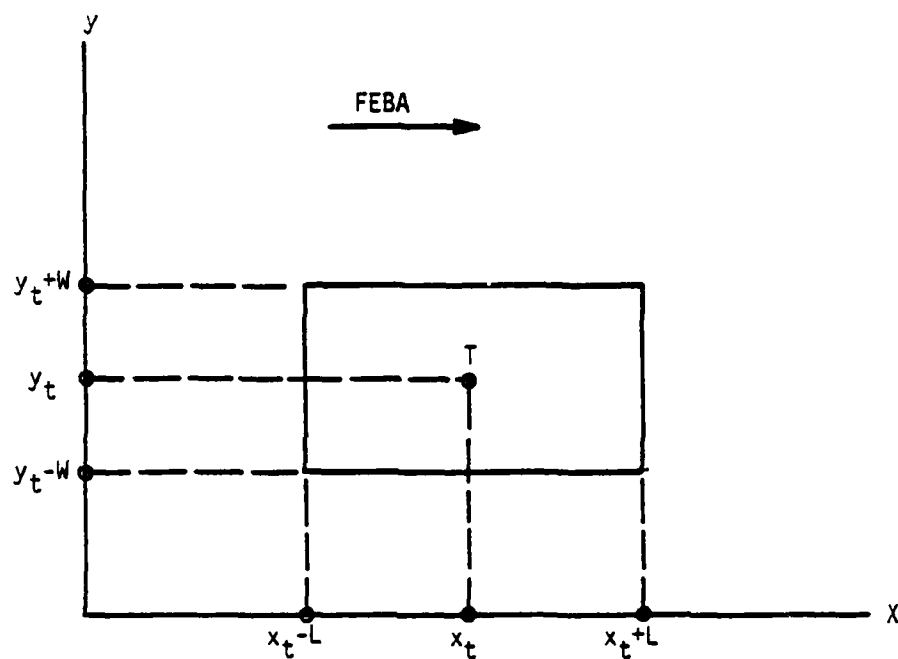


Figure 9. On the Probability of Occupancy for the Two-Dimensional Tank Array

The corresponding PDFs are

$$g_x(x) = \frac{1}{\sigma_x^2 \sqrt{2\pi}} \exp \left\{ -\frac{(x - \bar{x}(t))^2}{2\sigma_x^2} \right\} \quad (15-3a)$$

$$g_y(y) = \frac{1}{\sigma_y^2 \sqrt{2\pi}} \exp \left\{ -\frac{(y - \bar{y}(t))^2}{2\sigma_y^2} \right\} \quad (15-3b)$$

with

$$\sigma_x^2 = \sigma_0^2 + t^2 \sigma_v^2 \quad (15-4a)$$

$$\sigma_y^2 = \sigma_0^2 + t^2 \sigma_v^2 \quad (15-4b)$$

and

$$\bar{x}(t) = \langle x(t) \rangle = t\bar{v}_x \quad (15-5a)$$

$$\bar{y}(t) = \langle y(t) \rangle = t\bar{v}_y \quad (15-5b)$$

For the probability of occupancy, we turn to figure 9, where L and W denote the half-length and the half-width of the array. The figure shows a rectangle with the sides parallel to the coordinate axes. The side lengths in x and y -direction are $2L$ and $2W$, respectively. The center, T , is the contemplated target point. It has the coordinates x_t and y_t .

Necessary and sufficient for the occupancy of T is that the middle point of the array lies within the rectangle. Therefore, the probability of occupancy is

$$P_o(T) = G_x(x_t) G_y(y_t) \quad (15-6)$$

with

$$G_x(x_t) = \int_{x_t - L}^{x_t + L} g_x(x) dx \quad (15-7a)$$

$$G_y(y_t) = \int_{y_t - L}^{y_t + L} g_y(y) dy \quad (15-7b)$$

Finally, let w denote the "interior area density" of tanks. If the array contains N tanks, we have

$$w = \frac{N}{4LW} \quad (15-8)$$

For the "exterior area density", we then obtain

$$\omega(x_t, y_t) = w P_0(t) \quad (15-9)$$

With the functions $P_0(t)$ and $\omega(x_t, y_t)$, we have essentially provided the methodology for the two-dimensional problem.

APPENDIX

The purpose of this appendix is to demonstrate the validity of the approximation implied by equations (14-8) and (14-9). In order to save time and space, we prove the assertion only for the original SDF, $F(s)$, rather than the RSDF, $F_R(s)$. In the latter case, the proof would require the numerical solution of some integrals. However, it is to be expected that, if the approximation is valid for the SDF, it is valid also for the RSDF, though perhaps not to the same degree.

Using now the SDF instead of the RSDF, we write the assertion from section N in the following, more elaborate form:

$$I \stackrel{\text{Def}}{=} \int_{S_1 = -\infty}^{+\infty} \left\{ \int_{S_2 = -\infty}^{+\infty} \phi(S_1 + S_2) F(S_1) F(S_2) ds_2 \right\} ds_1 \quad \substack{(A-1) \\ S_1 = \text{const}}$$

$$J \stackrel{\text{Def}}{=} \frac{1}{\sigma_s \sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(2s) \text{Exp} \left\{ \frac{-(S - S_0)^2}{2\sigma_s^2} \right\} ds \quad (A-2)$$

$$I \simeq J \quad (A-3)$$

For the proof, we perform in equation (1) the substitution

$$S_1 + S_2 = 2S'_1 \quad (A-4)$$

$$S_1 - S_2 = 2S'_2$$

which is equivalent to

$$S'_1 + S'_2 = S_1 \quad (A-5)$$

$$S'_1 - S'_2 = S_2$$

The Jacobian is

$$\frac{\partial (S_1, S_2)}{\partial (S'_1, S'_2)} = -2 \quad (A-6)$$

Therefore, equation (1) assumes the form

$$I = -2 \int_{S'_1 = -\infty}^{+\infty} \left\{ \int_{S'_2 = +\infty}^{-\infty} \phi(2S'_1) F(S'_1 + S'_2) F(S'_1 - S'_2) ds'_2 \right\} ds'_1 \quad S'_1 = \text{const}$$

(A-7)

Now observing equation (13-14), we have

$$F(S'_1 + S'_2) F(S'_1 - S'_2) = \frac{1}{\sigma_s^2 2\pi} e^{-Q(S'_1, S'_2)} \quad (A-8)$$

with

$$Q(S'_1, S'_2) = \frac{(S'_1 + S'_2 - S_0)^2 + (S'_1 - S'_2 - S_0)^2}{2\sigma_S^2} \quad (A-9)$$

Here, Q can be brought into the following form

$$Q(S'_1, S'_2) = \frac{(S'_1 - S_0)^2 + S'^2_2}{\sigma_S^2} \quad (A-10)$$

If we heed equation (10) in equation (7), we see that the double-integral splits into two single integrals. We express this by writing

$$I = I_1 \cdot I_2 \quad (A-11)$$

where we still have some freedom in defining I_1 and I_2 . We also note that the subscripts and the prime at the symbols S'_1 and S'_2 are now dispensable. Dropping these insignia, we write

$$I_1 = \frac{2}{\sigma_S^2 2\pi} \int_{-\infty}^{+\infty} \phi(2S) \exp \left\{ -\frac{(S - S_0)^2}{\sigma_S^2} \right\} ds \quad (A-12)$$

$$I_2 = \int_{-\infty}^{+\infty} \exp \left\{ -\frac{S^2}{\sigma_S^2} \right\} ds \quad (A-13)$$

Here, I_2 is solvable in closed form. Using equations (4-4a), (4-4b), and (4-5a), we get

$$I_2 = \sigma_s \sqrt{\pi} \quad (\text{A-14})$$

If then we combine equations (11), (13), and (14), we obtain

$$I = \frac{1}{\sigma_s \sqrt{\pi}} \int_{-\infty}^{+\infty} \phi(2S) \exp \left\{ -\frac{(S-S_0)^2}{\sigma_s^2} \right\} ds \quad (\text{A-15})$$

This equation is still exact. It becomes an approximation if we replace σ_s by $\sigma_s \sqrt{2}$. In this case, equation (15) goes over into equation (2), that is, I is approximated by J .

It is now in order to say a few words about the reasons for and the accuracy of the approximation. The need for it can be seen from equations (14-12) and (14-13). In equation (14-12), the conditional probabilities of occupation for all columns are summed up. In equation (14-13), this sum is then integrated with one and the same RSDF, $F_R(S)$. This, of course, saves a lot of numerical integrations. But it introduces an error.

Turning now to the accuracy, we have to say a few words about the aimpoint selection. Clearly, the optimal aimpoint is always the point where the probability of occupancy has a maximum. This means that the first bomb should be aimed at the aggregate's middle point. At this point, the probability of occupancy is largest for column 3 (see figures 7a, 7b), somewhat smaller for columns 2 and 4, and still smaller for columns 1 and 5. However, the error introduced by the approximation refers to columns 1 and 5. In other words, the error refers to the smallest contributor to the probability of occupancy.

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